Quantum Ergodicity on Graphs

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Quantum Chaos vs Integrable Systems

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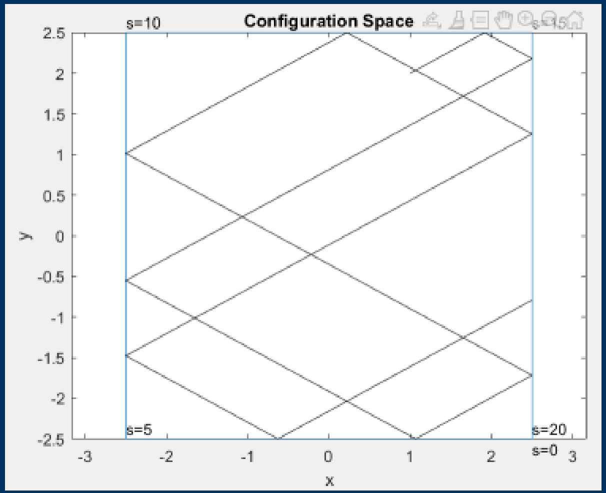
Integrable systems (lots of patterns). A slight change in direction does not greatly change the trajectory.



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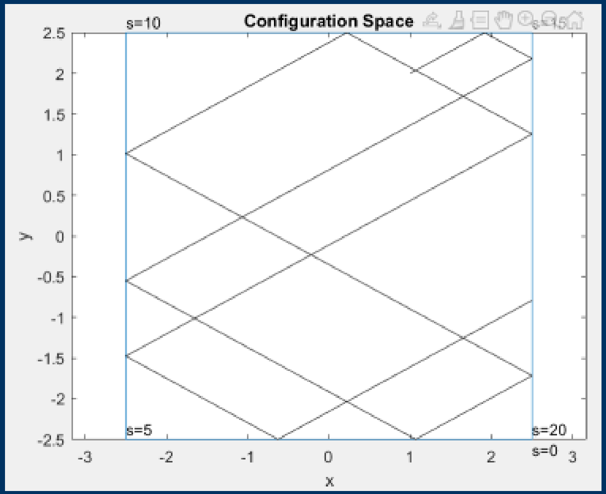
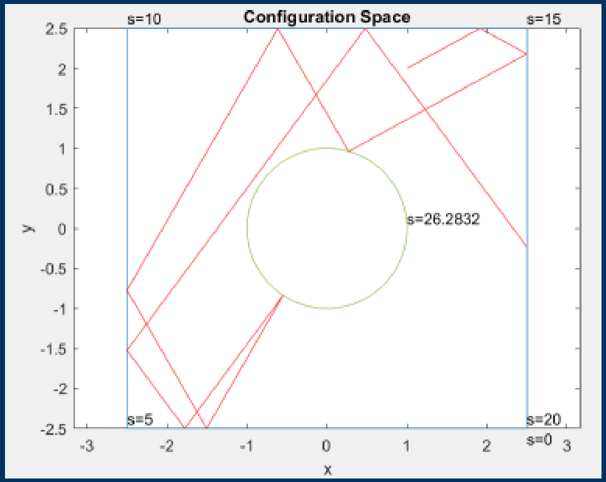
Integrable systems (lots of patterns). A slight change in direction does not greatly change the trajectory.

Chaotic Systems (no patterns). A slight change in coordinates leads to a vastly different path.





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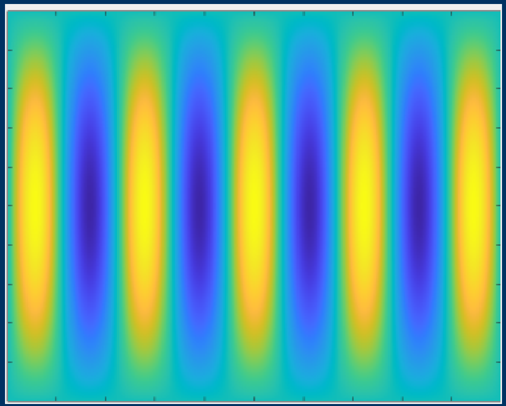

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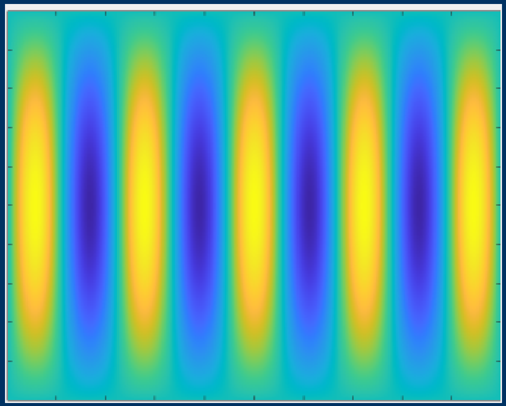
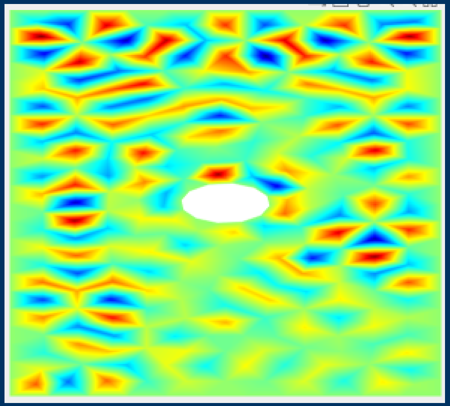
Integrable systems: We expect spectral fluctuations to be Poisson, and eigenfunctions to be localized in phase space.

Quantum chaotic systems: We expect spectral fluctuations to be those of large random matrices, and eigenfunctions to be equidistributed in phase space.





Behavior of Eigenfunctions

- [Shnirelman '74, Colin de Verdière '85, Zelditch '87] Quantum Ergodicity Theorem: A Riemannian manifold (M, g) with ergodic geodesic flow is such that almost all high energy eigenfunctions of the Laplacian are equidistributed.

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- Namely, for a compact Riemannian manifold (M, g) of volume 1, consider an ordered orthonormal basis of eigenfunctions $\{\phi_k\}_{k \in N}$. If geodesic flow is ergodic w.r.t. Liouville measure, then for any continuous test function a, we have

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{k, \lambda_k \leq \infty} \left| \langle \phi_k, a\phi_k \rangle - \int_M a(x) d\mathrm{Vol}(x) \right|^2 \to 0.$$

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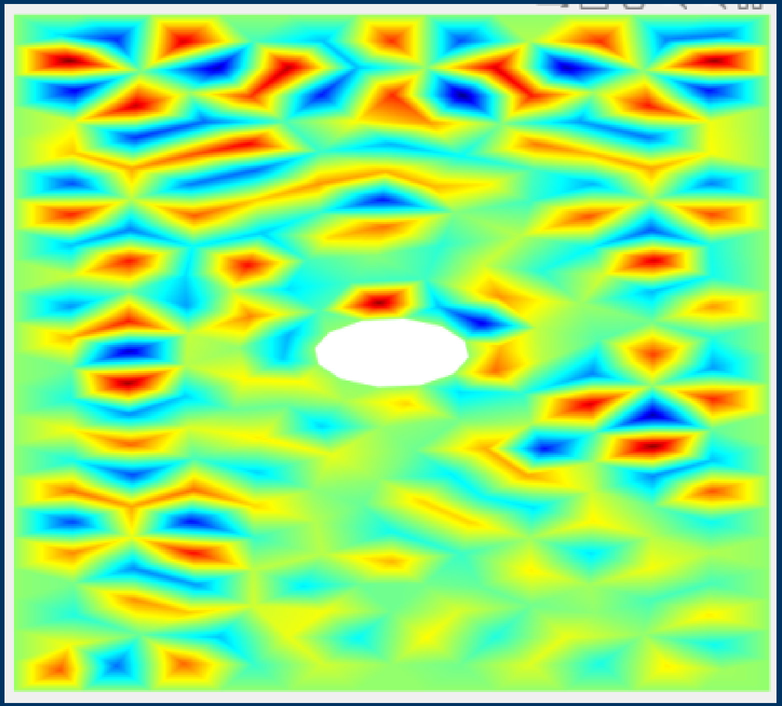


Quantum Unique Ergodicity

- [Rudnick-Sarnak] Quantum Unique Ergodicity Conjecture: With negative curvature,

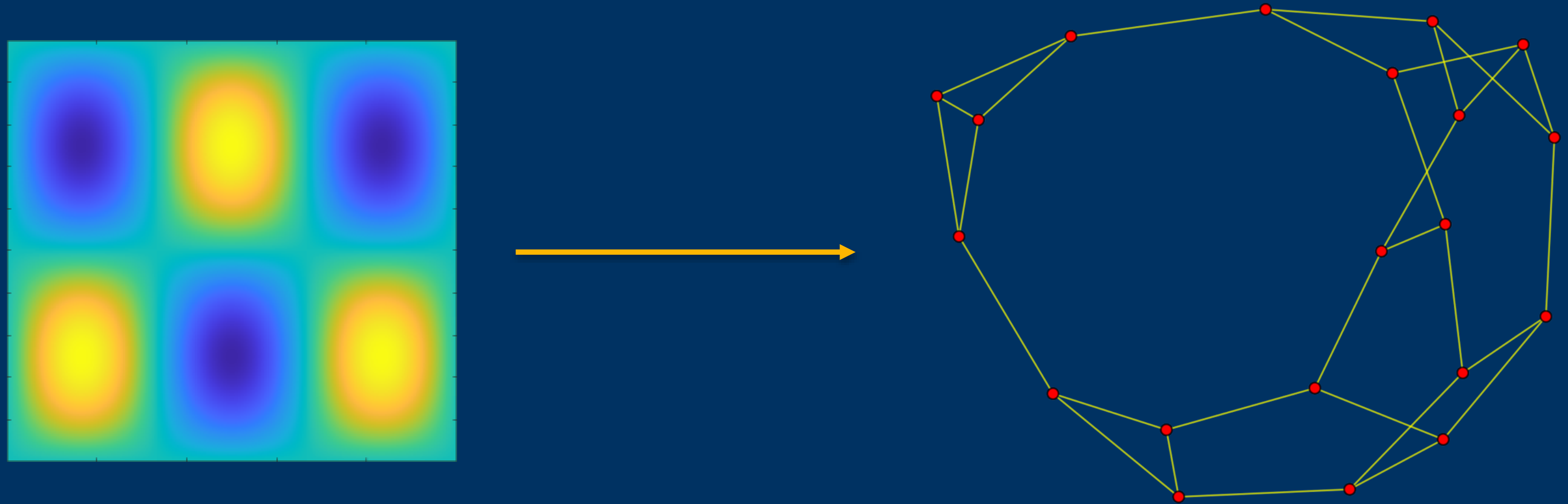
$$\left| \langle \phi_k, a\phi_k \rangle - \int_M a(x)\, d\mathrm{Vol}(x) \right|^2 \to 0$$

without averaging!

Discrete Graphs as a Model for Quantum Chaos

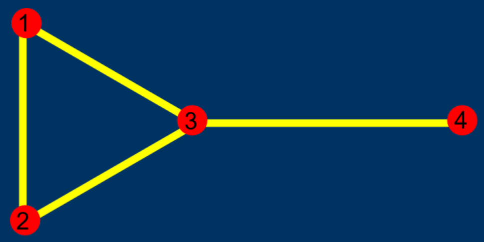
- [Kottos-Smilansky 1997,1999] Initiate using large regular graphs as a model for quantum chaos by examining the eigenvectors of the discrete Laplacian.

- Rather than take the high energy limit, we send the number of vertices to ∞.



Adjacency Matrix

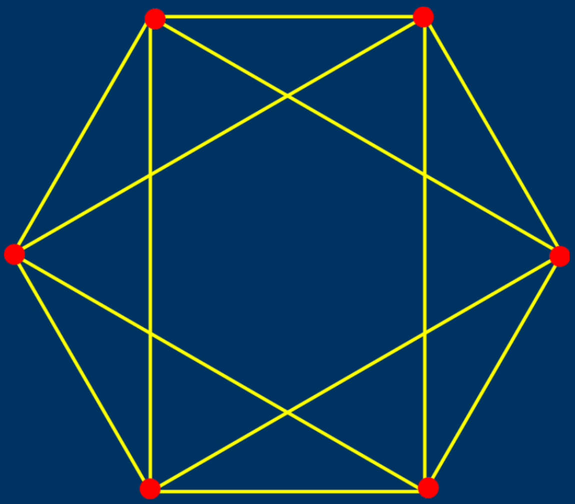
- Encode the walk through an "adjacency matrix" A, with rows/columns corresponding to the vertices, and putting a 1 between connected vertices.

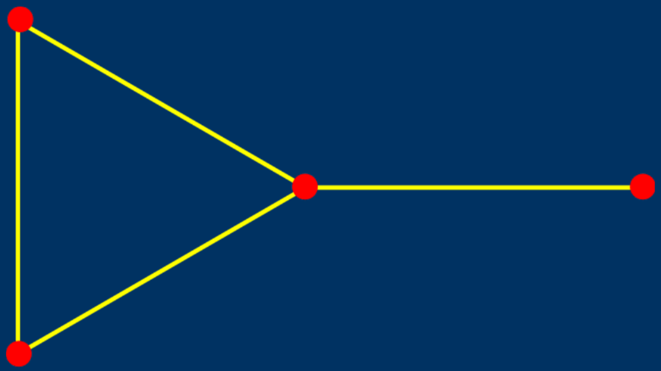


$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Note that as the matrix is symmetric, the eigenvalues are real and can be ordered $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, where n is the number of vertices.
- Multiplying by the matrix can be thought of as a step in the walk.
- The entry $(A^k)_{uv}$ counts to walks of length k between u and v.

Regularity





regular non-regular

- Often, for simplicity, we will assume our graph is regular, as it gives us our top eigenvector and eigenvalue.

Discrete Chaos

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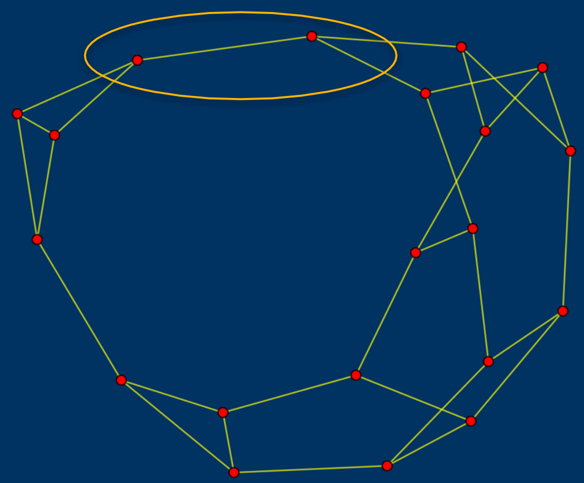
- New goal: show eigenvectors of graphs are equidistributed.

Discrete Chaos

- Recall that chaos means equidistribution of eigenfunctions.
- (Max Born) For a normalized eigenvector ψ, we think of $\psi(v)^2$ as a distribution of mass at energy level associated with λ.
- New goal: show eigenvectors of graphs are equidistributed.
- Equivalently: show that most of the mass of the eigenvector cannot be on a small number of entries.



Discrete Graphs as a Model for Quantum Chaos

- Define ψ_S to be the projection of the vector ψ onto coordinates S.

- [Brooks-Lindenstrauss '13] – For a normalized eigenvector ψ of the adjacency matrix, if my graph has no cycles of length less than k, then any set $S \in V$ such that $||\psi_S|| \geq \epsilon$ has $|S| \geq \epsilon^2 d^{c\epsilon^2 k}$.

- [Ganguly-Srivastava '21] – Improve this to if $||\psi_S|| \geq \epsilon$ has $|S| \geq \epsilon d^{c\epsilon k}$.

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- Consider an infinite family of graphs $(G_N)_{N\in\mathbb{N}}$ with corresponding families of eigenbases $\left(\{\phi_i^N\}\right)_{N\in\mathbb{N}}$. If $(G_N)_{N\in\mathbb{N}}$ are

1. Expanders
2. High girth

and if $a_N: V_N \to \mathbb{R}$ is uniformly bounded, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k\in[N]} \left| \langle \phi_k^N, a\phi_k^N \rangle - \int_V a(x) d\mathrm{Vol}(x) \right|^2 \to 0$$

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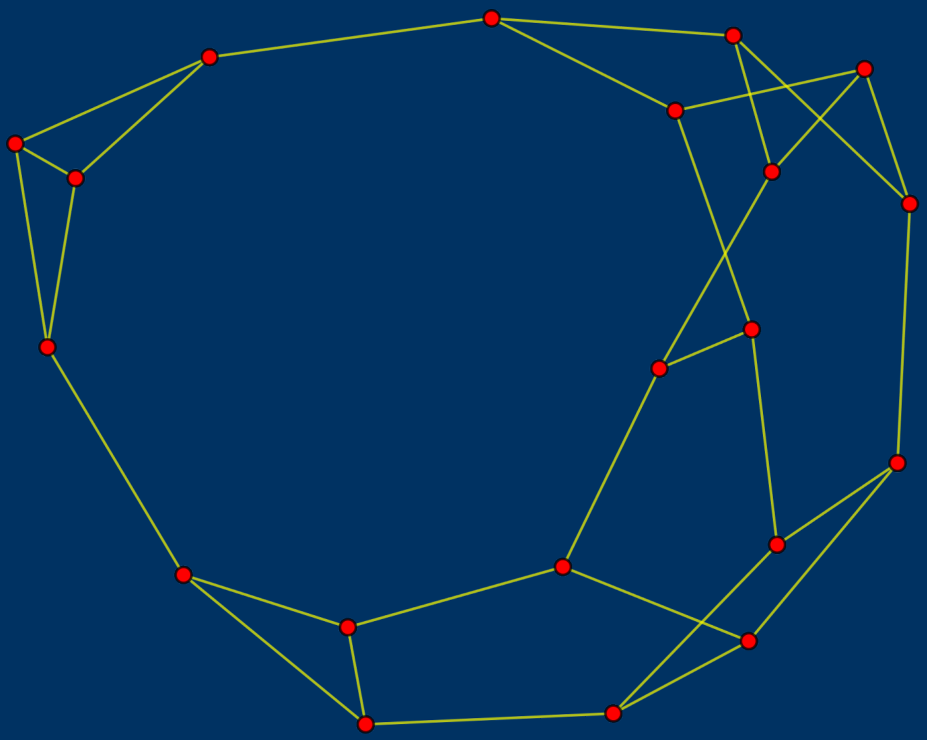

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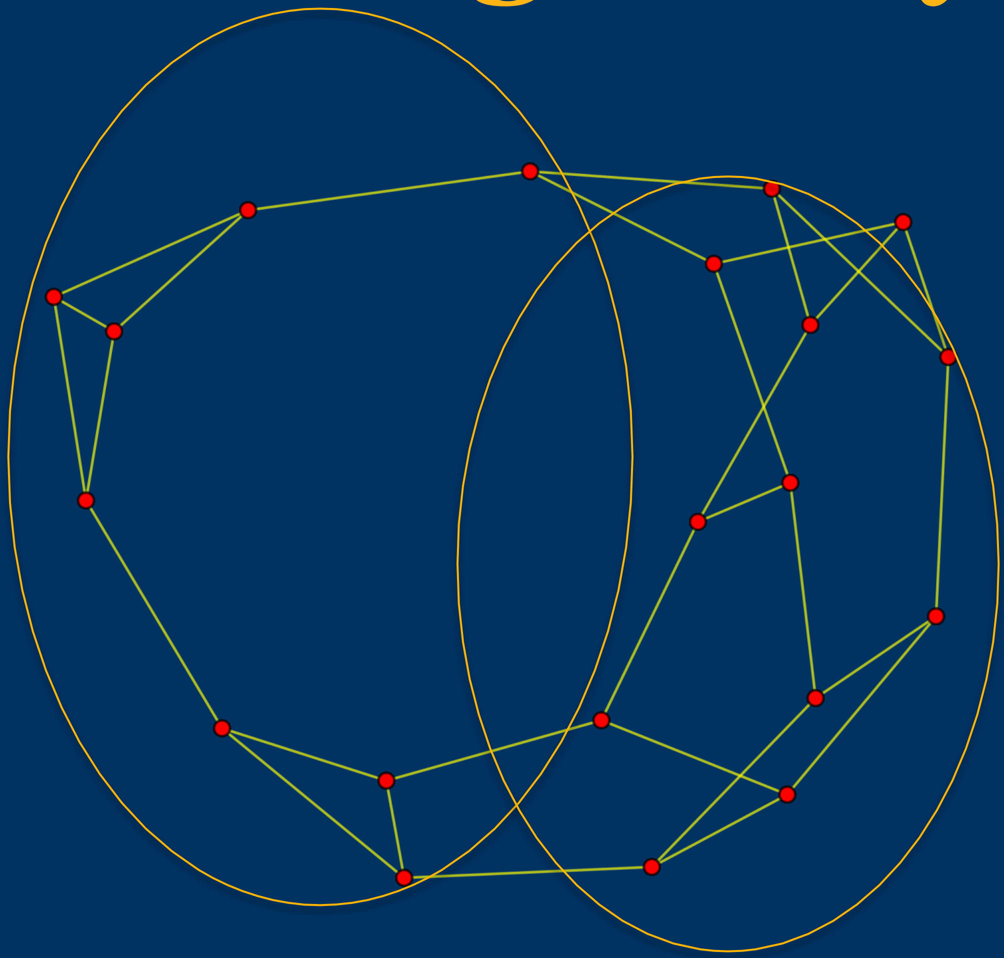

- Any partition divides almost every eigenvector almost evenly

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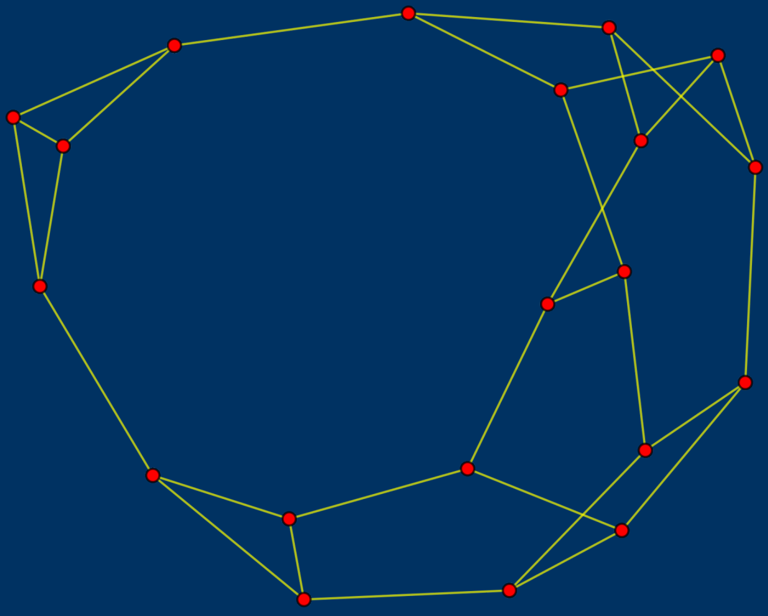
Expansion

- A regular graph is an **expander** if all of its nontrivial eigenvalues have absolute value at most $(1-\epsilon)d$.



Random Walks

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- Because of it being used as the rapidity of the random walk, expansion is key to Markov Chain Monte Carlo and other algorithms.

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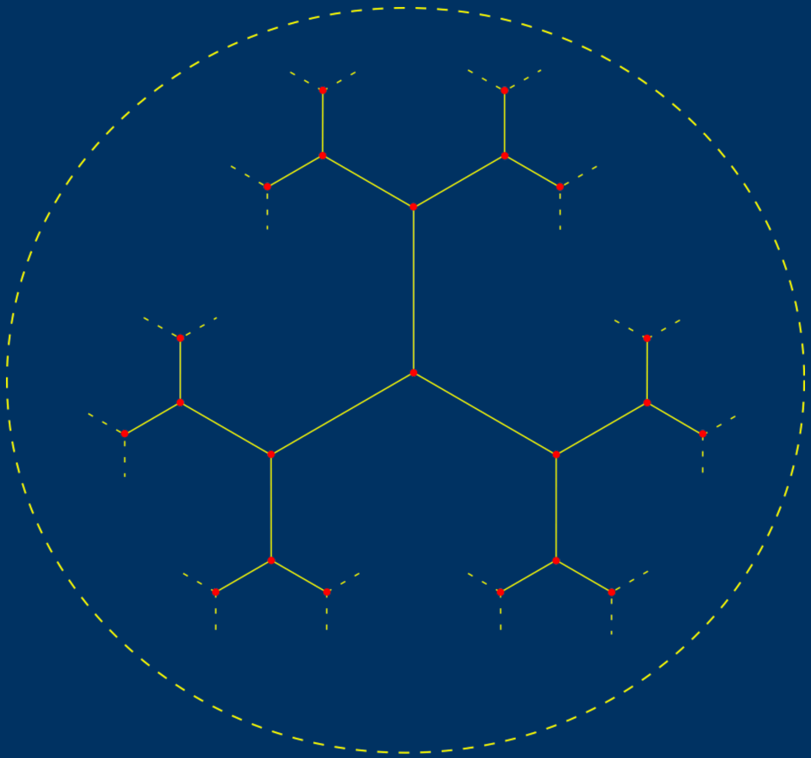
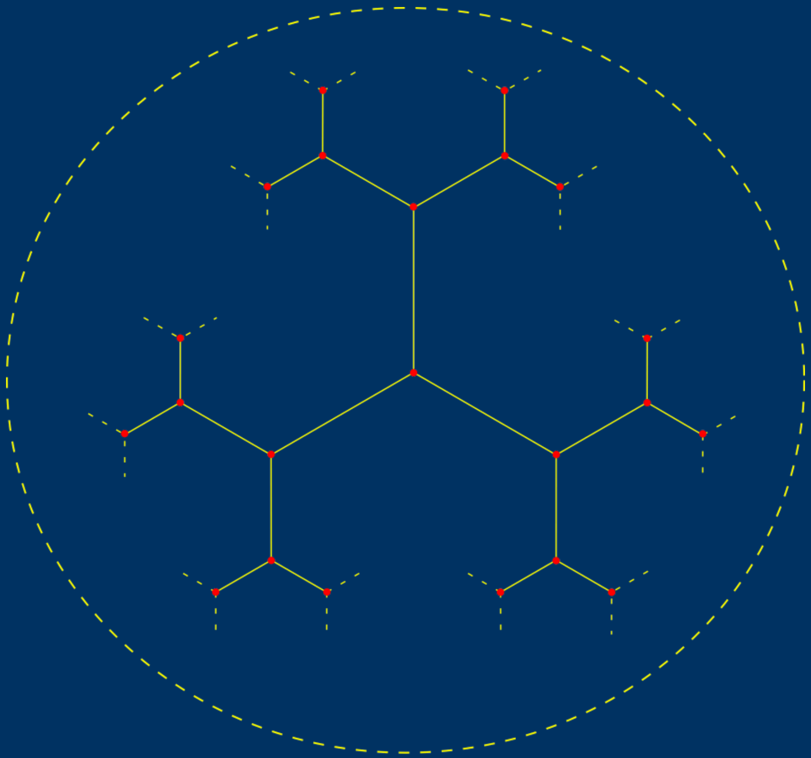
Girth

- There are no short cycles
- Same condition as was seen in the previous results (Brooks-Lindenstrauss, Ganguly-Srivastava)



Girth

- Walks mix optimally quickly on a **local scale**.



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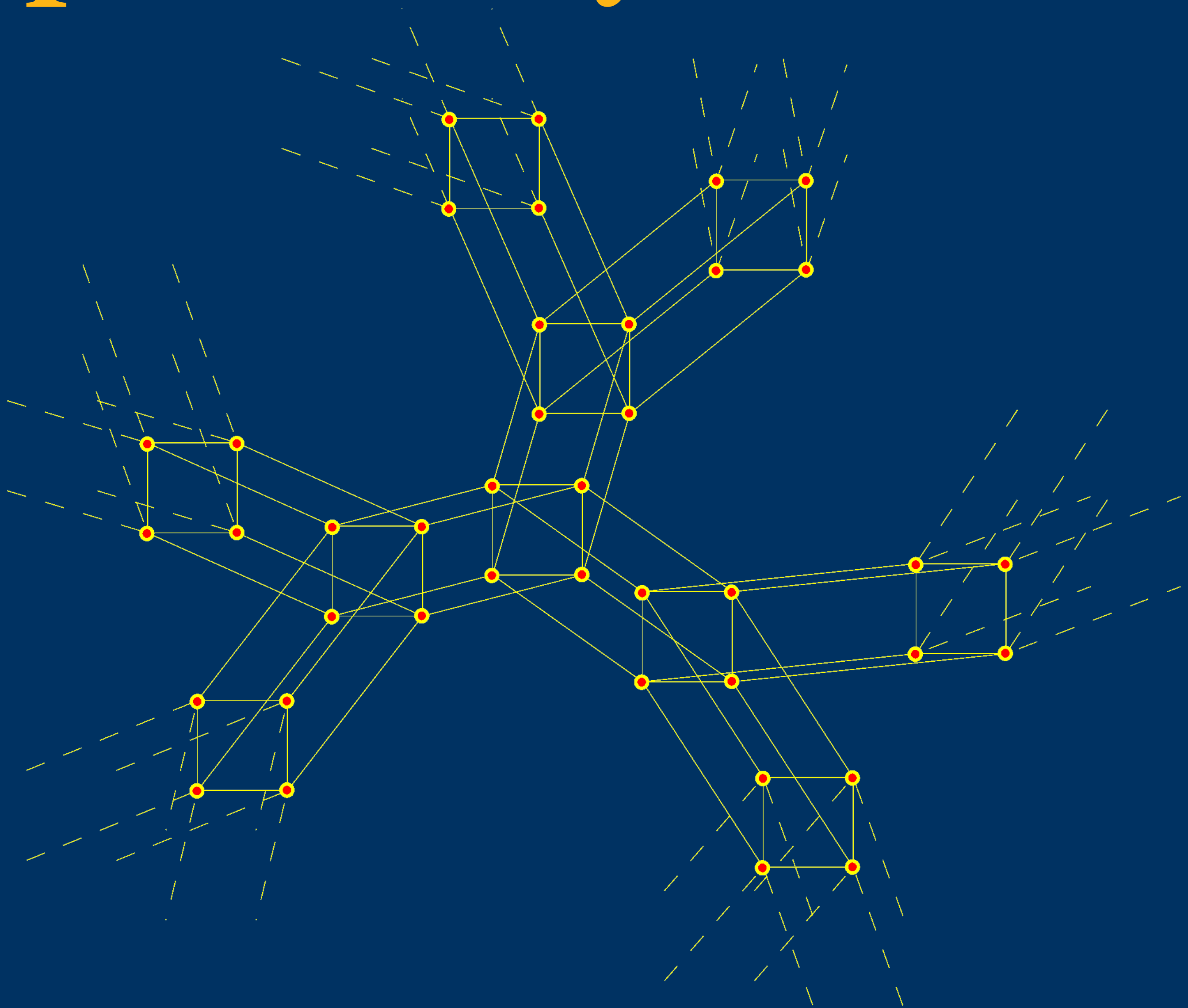
- Spectral gap on a manifold gives rate of exponential mixing of geodesic flow.
- As ergodicity is the only requirement in Shnirelman’s theorem, can we remove the girth requirement of the discrete version?
- **No!**

Expansion by itself

- [M ‘22] There is an infinite family of graphs that satisfy expansion but are not high girth, and violate quantum ergodicity.

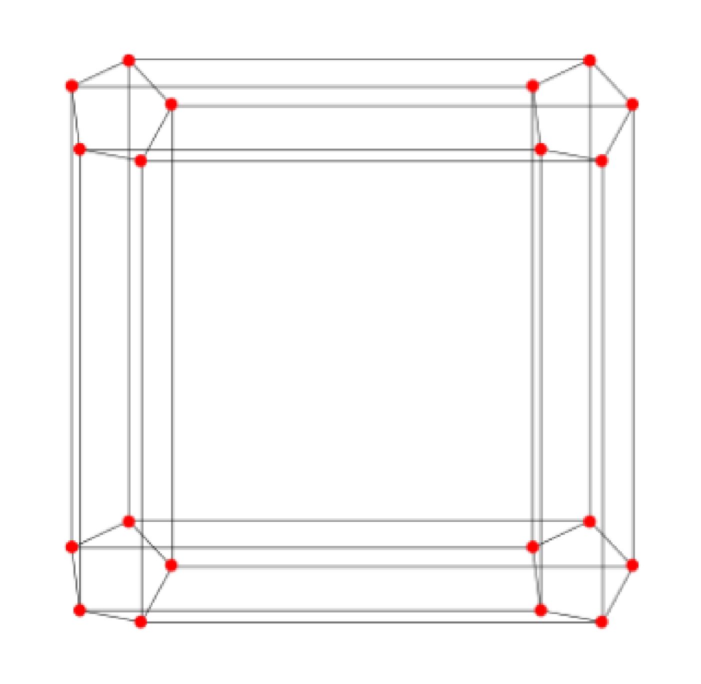
- Namely, we can partition the vertices into two sets, such that many eigenvectors are uneven across these sets.

Expansion by itself



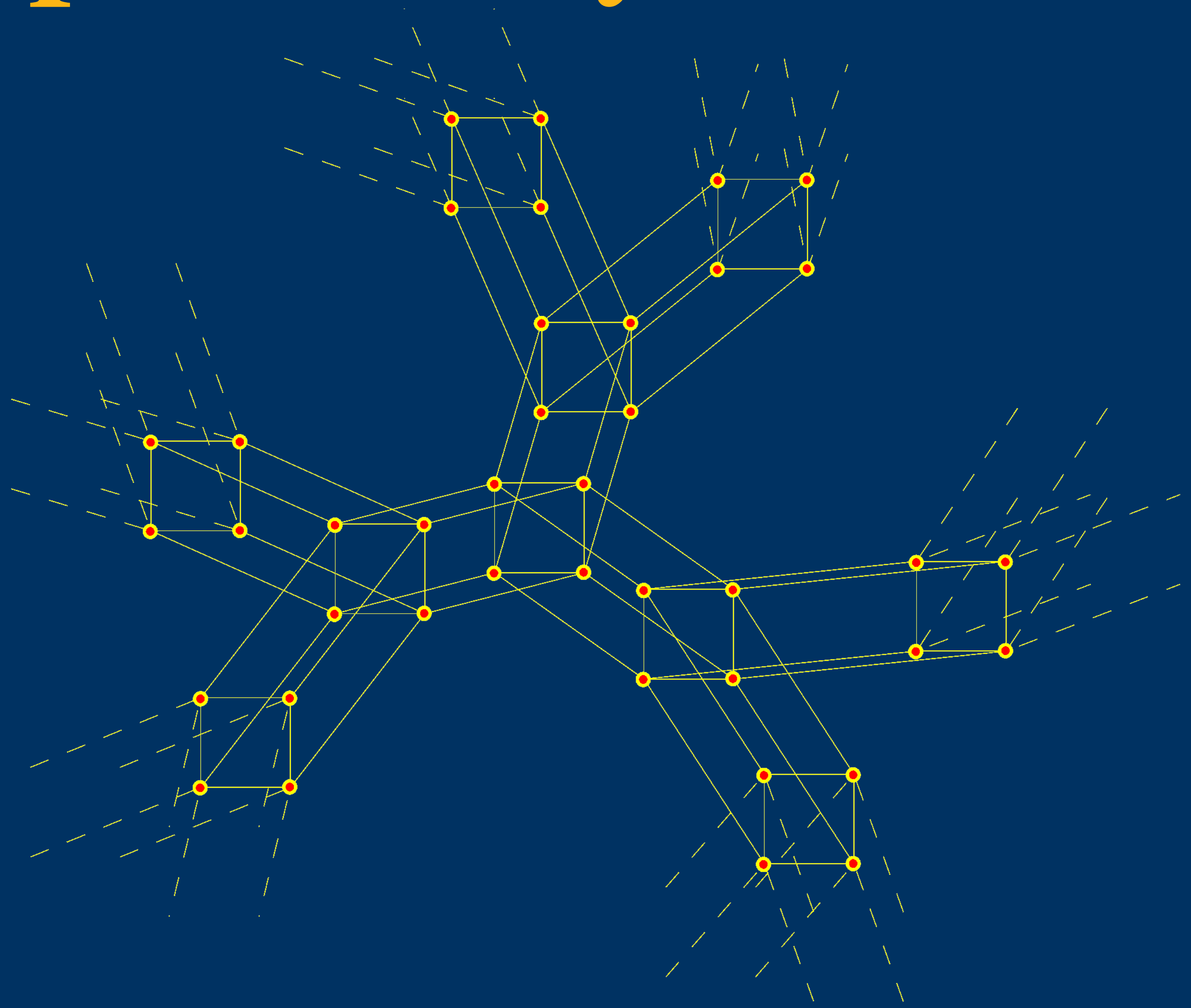
Expansion by itself

- The Cartesian product expands each vertex into a copy of a graph.



- Because of the nature of the Cartesian product, and because the square has localized eigenvectors, this larger graph also has localized eigenvectors (the same localization).

Expansion by itself



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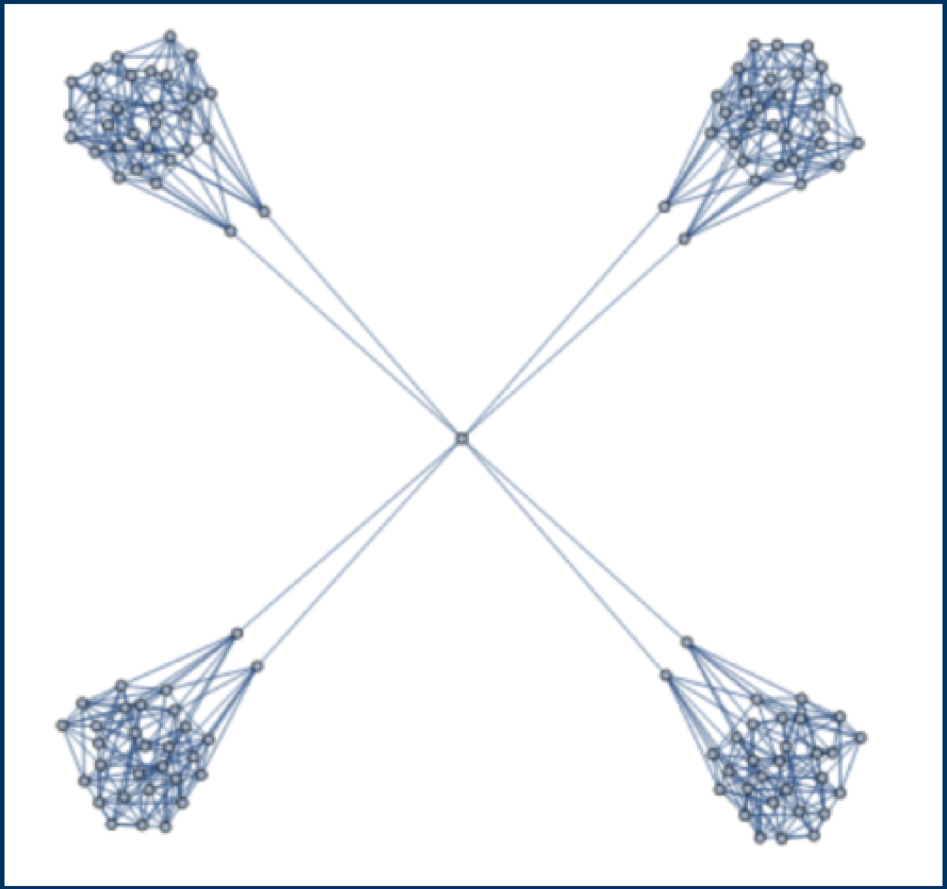

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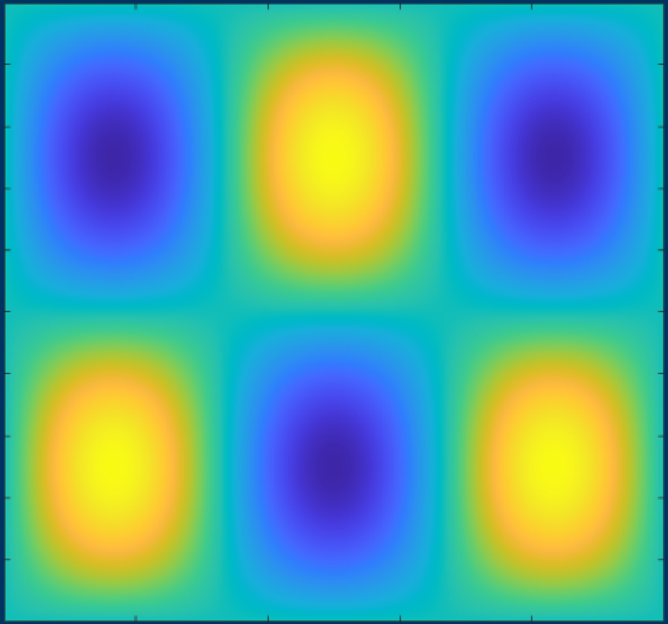
- Perhaps to have ergodicity of geodesic flow is not equivalent to expansion, we must consider it on a local scale as well, which is given by girth.
- Without one of expansion or girth, we can create large scale **patterns** that we can take advantage of in the eigenvector.
- That doesn't seem very quantum chaotic!
- If my test function avoids patterns, then perhaps the statement will still be true.

Other Delocalization

- The beauty of quantum ergodicity is the generality of the model in which it is true.
- Stronger delocalization results are true for more general models, but the hope is still to push past these barriers.

Courant's Nodal Domain Theorem

- [Courant] The zero set of the kth smallest Dirichlet eigenfunction of the Laplacian on a smooth bounded domain in $\mathbb{R}^d$ partitions it into at most k components.
- These components, known as nodal domains, have garnered significant attention in spectral geometry and mathematical physics.



A heat map of the 6th Dirichlet eigenfunction of the square.

Discrete Version

- The **nodal domains** of a vector f on the vertices of a graph G are the maximal connected components of all the same sign.

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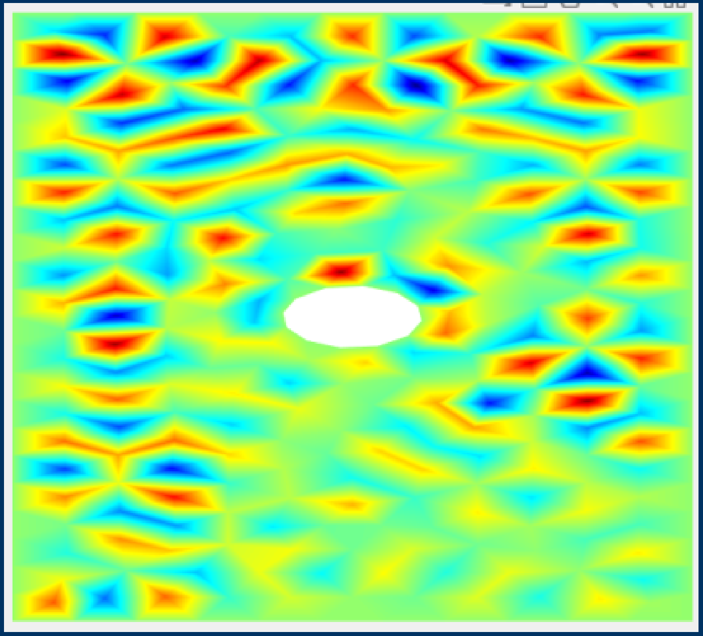
Berry's Conjecture to Many Nodal Domains

- **Claim**: In both continuous and discrete space, having many nodal domains is chaotic behavoir.



Result

[Ganguly-M-Mohanty-Srivastava] Fix $d \geq 3$ and $\alpha > 0$. Then with probability $1 - o_n(1)$, every eigenvector of the adjacency matrix of a $G(n,d)$ sampled graph with eigenvalue $\lambda \leq -2\sqrt{d-2} - \alpha$ has $\Omega(n/\text{polylog}(n))$ nodal domains.

Outline

- We split into cases based on whether the eigenvector is **localized** or **delocalized** (whether the mass of the eigenvector is well spread or not).

- **Definition:** an eigenvector ψ is **delocalized** if for fixed $\epsilon, \delta > 0$,
$$|\{v \in V | \psi^2(v) \geq \epsilon/n\}| \geq \delta n.$$

- If the eigenvector is delocalized, we can use the proximity of an eigenvector of a random regular graph to a Gaussian distribution.

- If the eigenvector is localized, then we can argue using the local structure of random regular graphs.

Future directions

- Perhaps we can similarly treat ℓ_2 delocalization vs quantum ergodicity.

- The interplay between results in continuous and discrete space remains fascinating, and not fully understood. Perhaps these techniques can shed light on the properties of manifolds.

Thank you!